3.7/3.8: Mass-Spring Summary

m = mass attached to end of spring
γ = damping constant
k = spring constant
F(t) = external force function
u(t) = displacement from rest at time *t*

We derived that

$$mu'' + \gamma u' + ku = F(t)$$

We consider four situations:

In 3.7

Case 1: No damping, no forcing.
Case 2: Damping, no forcing.

In 3.8

Case 3: No damping, with forcing.
Case 4: Damping with forcing.

<u>*Case 1*: F(t) = 0 and $\gamma = 0$</u>

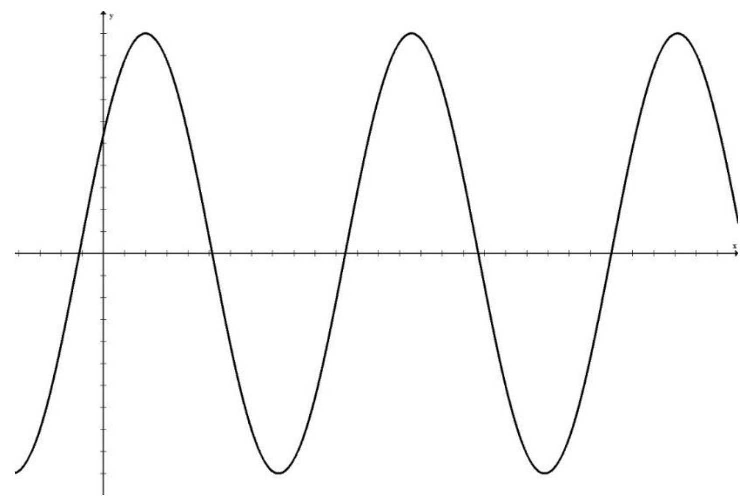
$$mu'' + ku = 0$$

$$mr^2 + k = 0 \text{ gives } r = \pm\sqrt{k/m}\ i$$

Soln: $u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$

$$\omega_0 = \sqrt{k/m} = \boldsymbol{natural}\ freq.$$

$$R = \sqrt{c_1^2 + c_2^2} = amplitude.$$



***Case 2*: F(t) = 0 and $\gamma > 0$**

$$mu'' + \gamma u' + ku = 0$$

$$mr^2 + \gamma r + k = 0 \text{ gives}$$

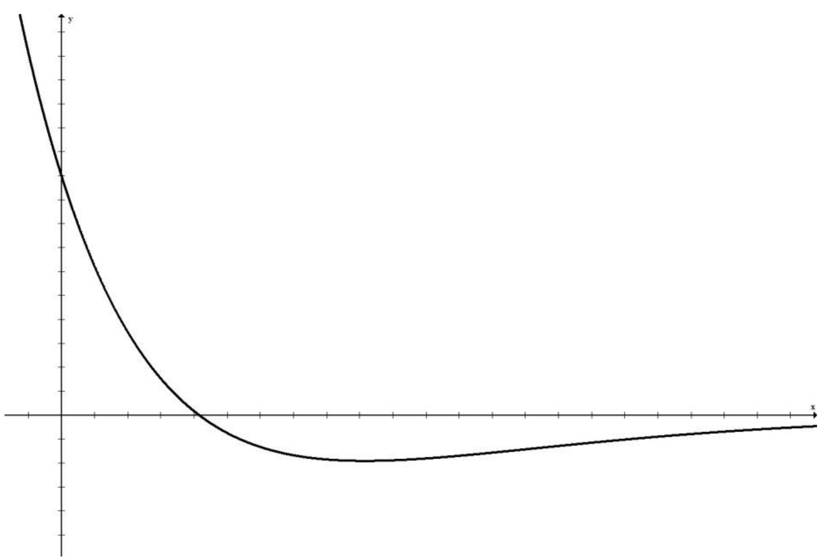
$$r = -\frac{\gamma}{2m} \pm \frac{1}{2m}\sqrt{\gamma^2 - 4mk}$$

2a: $\gamma > 2\sqrt{mk}$, **overdamped**

Soln: $u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

2b: $\gamma = 2\sqrt{mk}$, **critically damped**

Soln: $u(t) = c_1 e^{rt} + c_2 t e^{rt}$



2c: $\gamma < 2\sqrt{mk}$, **underdamped**

$$r = -\frac{\gamma}{2m} \pm \frac{1}{2m}\sqrt{4mk - \gamma^2}\, i$$

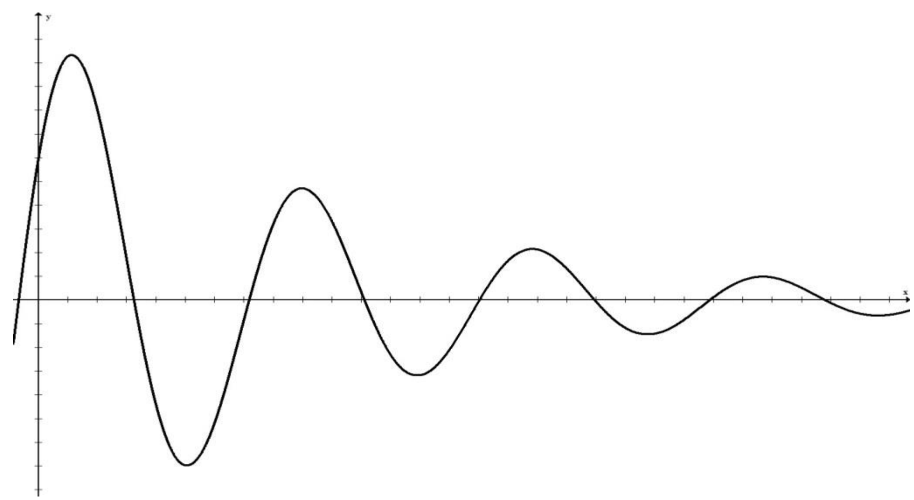
Soln: $u(t) = e^{\lambda t}(c_1 \cos(\mu t) + c_2 \sin(\mu t))$

$$\lambda = -\frac{\gamma}{2m}$$

$$\mu = \frac{1}{2m}\sqrt{4mk - \gamma^2} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}$$

= **quasi**-frequency

Note: As $\gamma \to 0$, $\mu \to \omega_0$



Case 3: $\boldsymbol{\gamma = 0, F(t) = F_0 \cos(\omega t)}$

$$mu'' + ku = F_0 \cos(\omega t)$$

Homogeneous solution

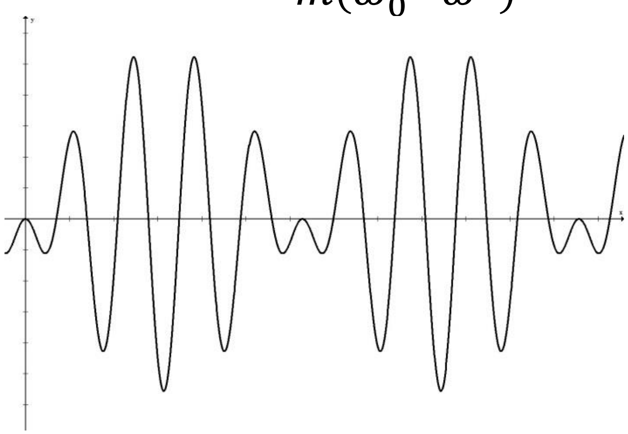
$$c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

General solution

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + u_p(t)$$

3a: If $\omega \neq \omega_0$, then use

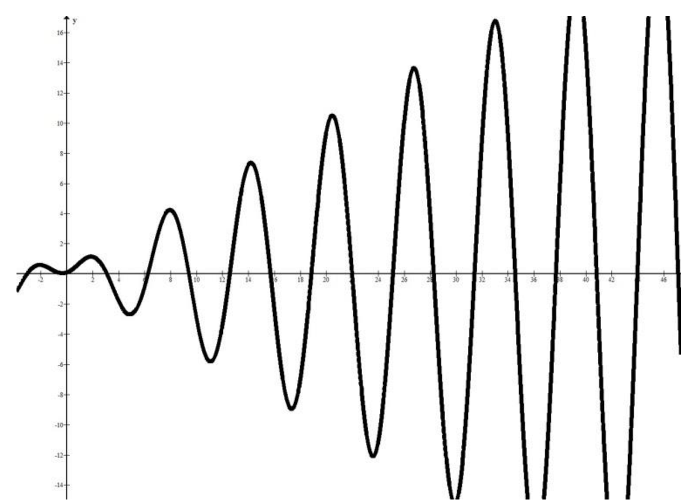
$$u_p(t) = A \cos(\omega t) + B \sin(\omega t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$



Aside: Picture is soln to
u''+16u = cos(5t), u(0)=0, u'(0)=0

3b: If $\omega = \omega_0$, then use

$$u_p(t) = At \cos(\omega t) + Bt \sin(\omega t) = \frac{F_0}{2m\omega_0} t \sin(\omega t)$$



Resonance!

Case 4: $F(t) = F_0 \cos(\omega t)\ and\ \gamma > 0$

Sol′n: $u(t) = u_c(t) + u_p(t)$

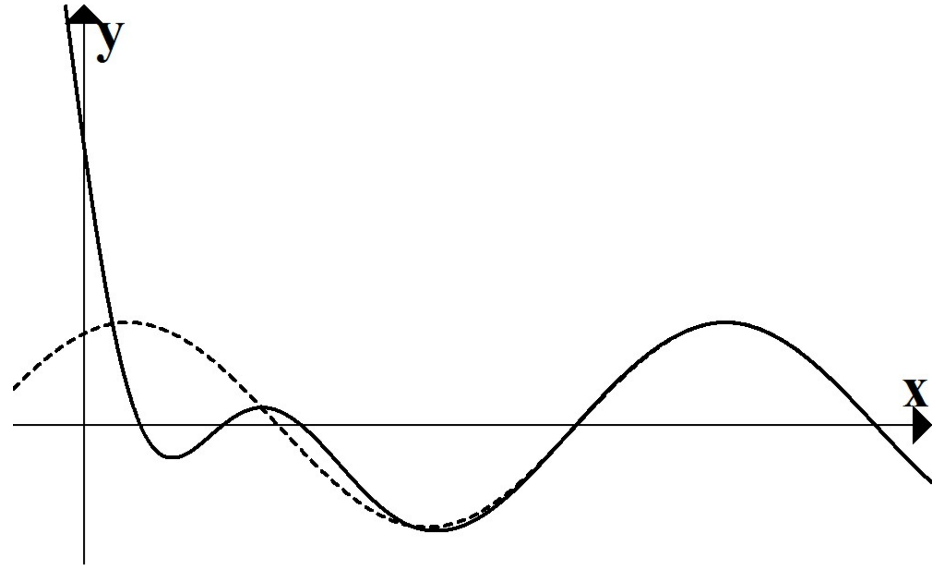
$u_c(t)$ =homogeneous sol'n

= ***transient*** sol'n

$u_p(t)$ =particular sol'n

= ***steady state*** sol'n

(also called ***forced response***)



Example: The solution to

$$u'' + 2u' + 5u = 10\cos(t)$$

u(0) = 6 and u'(0) = -11

looks like the solid graphed function in the picture, the steady state solution is the dotted solution.

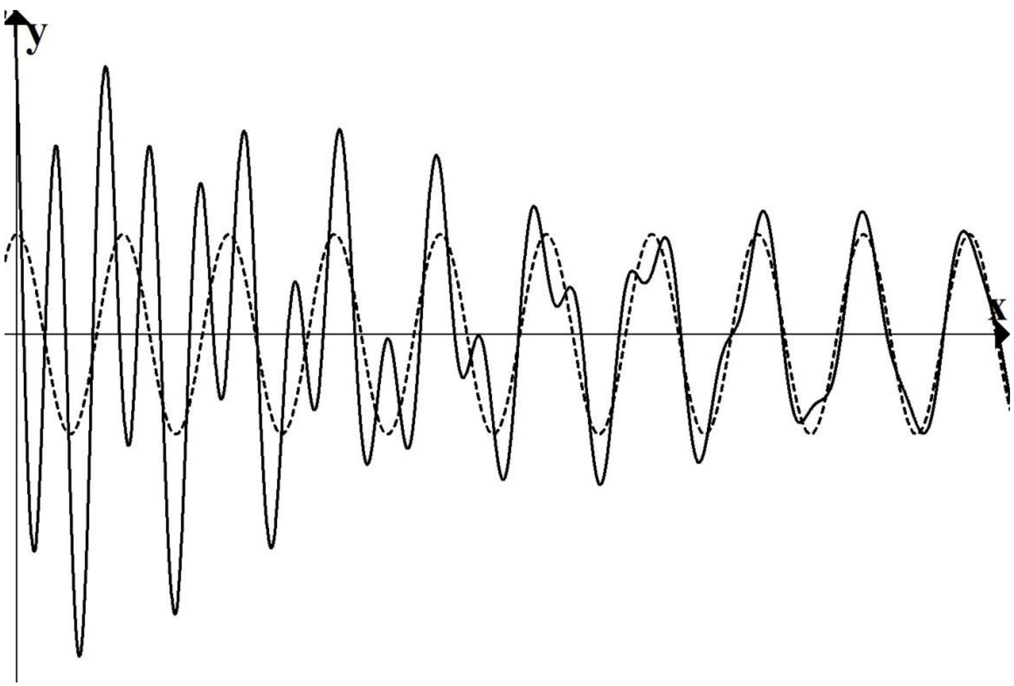
Solution:

u(t) = $e^{-t}(c_1 \cos(2t) + c_2 \sin(2t)) + 2\cos(t) + \sin(t)$

$= e^{-t}(4\cos(2t) - 6\sin(2t)) + 2\cos(t) + \sin(t)$

Example: Same problem with much smaller damping:

$$u'' + 0.1u' + 5u = 10\cos(t)$$



Some First Observations:

1. When there is damping, there is a *transient* part of the solution that always dies out.

2. If damping is smaller, it takes longer to die out.

3. The amplitude of the *steady state* solution dependents on m, γ, k, and F_0 in some way.

Solution:

u(t) $= e^{-0.05t}(c_1\cos(\mu t) + c_2\sin(\mu t)) + 2.4984\cos(t) + 0.0624\sin(t)$

Studying Amplitude of Steady State Soln

Consider the example

$$u'' + \gamma u' + 5u = 10\cos(\omega t)$$

Homogenous Solution:

The characteristic equation is

$r^2 + \gamma r + 5 = 0$

$r = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma^2 - 20}$

If $\gamma < \sqrt{20}$ (underdamped), then

$$\mu = \frac{1}{2}\sqrt{20 - \gamma^2} = \sqrt{5 - \frac{\gamma^2}{4}}$$

And Note: $w_0 = \sqrt{5}$

Particular Solution:

Using: $u_p(t) = A\cos(\omega t) + B\sin(\omega t)$

we get

$$(5 - \omega^2)A + \gamma\omega B = 10$$
$$-\gamma\omega A + (5 - \omega^2)B = 0$$

So (through some algebra):

$$A = \frac{10(5 - \omega^2)}{\omega^4 + (\gamma^2 - 10)\omega^2 + 25}$$

$$B = \frac{10\gamma\omega}{\omega^4 + (\gamma^2 - 10)\omega^2 + 25}$$

as you can see it starts to get messy.

u(t) $= e^{-\frac{\gamma}{2}t}(c_1\cos(\mu t) + c_2\sin(\mu t)) + A\cos(\omega t) + B\sin(\omega t)$

Let's look at the case when

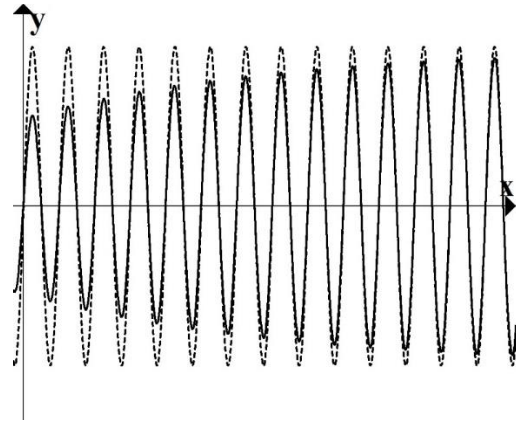
$$\omega = \sqrt{5}$$

$$u'' + \gamma u' + 5u = 10\cos(\sqrt{5}\,t)$$

then A = 0 and $B = \frac{10\gamma\sqrt{5}}{25+(\gamma^2-10)5+25} = \frac{2\sqrt{5}}{\gamma}$

$$u_p(t) = \frac{2\sqrt{5}}{\gamma}\sin(\sqrt{5}t)$$

γ	R
10	0.447
1	4.47
0.1	44.72
0.01	447.21
0.001	4472.14



Some Second Observations:

1. If the forcing frequency is close to the natural frequency, then tend to get large amplitude solutions.

2. In this case, the amplitude gets larger and larger the closer the damping is to zero.

General Discussion

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

Note: $\omega_0 = \sqrt{\frac{k}{m}}$, $\mu = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}$

Particular Solution:

$$u_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

Leads to

$$-\gamma\omega A + (k - m\omega^2)B = 0$$

$$(k - m\omega^2)A + \gamma\omega B = \mathrm{F}_0$$

The formulas for A and B are large to write out.

The amplitude of the steady state solution simplifies to:

$$R = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + \gamma^2\omega^2}}$$

Thinking of this as a function of ω the maximum steady state amplitude occurs when:

$$\omega_{max} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{2m^2}}$$

In particular, for small values of γ if $\omega \approx \omega_0$, then $R \approx \frac{F_0}{\gamma\omega}$ is large. (resonance)